Cauchy-Riemann and z^n is Analytic

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Abstract

Herein we use the Cauchy-Riemann equations and induction to show that z^n is analytic.

1 Introduction

The point of this note is to get some practice using the Cauchy-Riemann equations and mathematical induction to prove that z^n, for n a positive integer, is analytic. Knowledge of the Cauchy-Riemann equations is assumed.

Let $f(z) = u + iv$, where $z = x + iy$, and $u = u(x, y)$ is the real part of $f(z)$ and $v = v(x, y)$ is the imaginary part of $f(z)$. Then, Cauchy-Riemann equations are given as

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}\,, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}\,. \end{aligned} \tag{1}$$

where u and v are real-differentiable at a point in an open subset of $\mathbb{C}$.

2 Proof that z^n is analytic

In the case of $f(z) = z^n$, u and v will be polynomials in real variables x and y, and thus will be analytic.

The outline the proof is as follows: First, we show that $f(z)$ is analytic for the base case, $n = 1$. Next, we assume the Cauchy-Riemann (CR) equations hold for the case of $n = k$, and then show that for case $n = k + 1$ the Cauchy-Riemann equation also hold.

Base Case: $f(z) = x + iy$. Plugging into the CR equations (1), we get:

$$\begin{aligned} \frac{\partial x}{\partial x} &= 1 = \frac{\partial y}{\partial y}\,, \\ \frac{\partial x}{\partial y} &= 0 = -\frac{\partial y}{\partial x}\,. \end{aligned} \tag{2}$$

Now, we assume that the CR equations hold for

$$z^k = \alpha + i\beta \tag{3}$$

and then show that they hold for

$$z^{k+1} = (\alpha + i\beta)(x + iy) = (x\alpha - y\beta) + i(y\alpha + x\beta)\,, \tag{4}$$

where α and β are polynomials in where x and y — the exact nature of which we do not need to know.

The assumption that CR equations hold for (3) implies that

$$\frac{\partial \alpha}{\partial x} = \frac{\partial \beta}{\partial y}\,,$$
$$\frac{\partial \alpha}{\partial y} = -\frac{\partial \beta}{\partial x}\,. \tag{5}$$

So, to complete the proof, we need to use set $u = x\alpha - y\beta$ and $v = y\alpha + x\beta$, with the aid of (5) to show that the CR equations hold.

$$\begin{aligned}\frac{\partial u}{\partial x} &= \frac{\partial (x\alpha - y\beta)}{\partial x} \\ &= \alpha + x\frac{\partial \alpha}{\partial x} - y\frac{\partial \beta}{\partial x} \\ &= \alpha + x\frac{\partial \beta}{\partial y} + y\frac{\partial \alpha}{\partial y}\,, \\ &= \frac{\partial v}{\partial y}\,, \quad \checkmark \end{aligned} \tag{6}$$

which confirms the first CR equation; and

$$\begin{aligned}\frac{\partial u}{\partial y} &= \frac{\partial (x\alpha - y\beta)}{\partial y} \\ &= x\frac{\partial \alpha}{\partial y} - \beta - y\frac{\partial \beta}{\partial y} \\ &= -x\frac{\partial \beta}{\partial x} - \beta - y\frac{\partial \alpha}{\partial x} \\ &= -\frac{\partial v}{\partial x}\,, \qquad \checkmark \end{aligned} \tag{7}$$

which confirms the second CR equation.

Thus, by use of mathematical induction and the the Cauchy-Riemann equations, we have shown that z^n is an analytic function of z.